Anuran Makur

CALCULUS OF VARIATIONS TUTORIAL:

(adapted from "Mathematics and Technology" by Rousseau and Saint-Aubin & John Strain's notes)

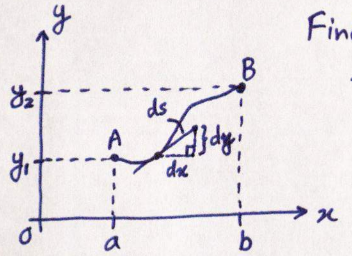
- **Introduction:**
  - branch of applied mathematics dealing with optimization over function spaces
  - many applications to physics & engineering
  - Used in Hamiltonian mechanics — bridge between Newtonian and quantum mechanics
  - Recall Lagrange multiplier method:

$$f: \mathbb{R}^n \to \mathbb{R}, \quad g: \mathbb{R}^n \to \mathbb{R}$$

$$\left.\begin{array}{ll} \min\limits_{x} & f(x) \\ \text{s.t.} & g(x) = c \end{array}\right\} \quad \underbrace{\mathcal{L}(x,\lambda)}_{\text{Lagrangian}} = f(x) + \lambda\big(g(x) - c\big), \qquad \underbrace{\nabla \mathcal{L} = 0}_{\text{stationary conditions}}$$

In variational calculus, we optimize over function spaces rather than $\mathbb{R}^n$.

- **Example 1:** (Shortest Path)



Find shortest path between A & B.
- Ans: Straight line $\rightarrow \Delta$ inequality.

Formalism: Let $y = y(x) \Rightarrow$ Path parametrized by $(x, y(x))$, $x \in [a,b]$.

Let $\underbrace{I[y]}_{\substack{\text{functional} \\ \text{(function of function)}}}$ = length of path between A & B

$$I[y] \triangleq \int_A^B ds = \int_A^B \underbrace{\sqrt{dx^2 + dy^2}}_{\text{Pythagoras' Thm}} = \int_a^b \sqrt{1 + (y')^2}\, dx$$

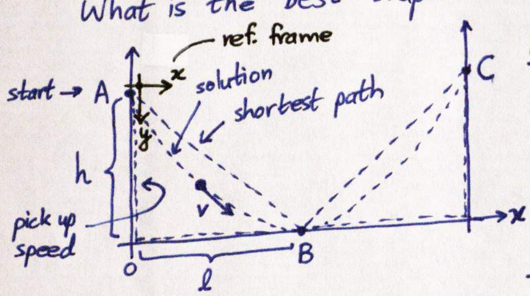
$\therefore$ Our problem is:

$$\min_{\substack{y(x): \\ y(a) = y_1,\, y(b) = y_2}} I[y] = \int_a^b \sqrt{1 + (y')^2}\, dx \; .$$

} can try to solve this $\rightarrow$ DIRECT METHOD

- **Example 2:** (Brachistochrone – "shortest time") $\rightarrow$ posed by Johann Bernoulli as contest & solved by Newton, Leibniz, L'Hôpital, Johann & Jacob Bernoulli

What is the best "shape" of a skateboard ramp?



Want: Minimum time from A to B, powered only by gravity.

Let the path between A & B be $(x, y(x))$. Let $I[y]$ = total time.

Formalism: (Conservation of energy)

Let energy at A be $E = 0$ (stationary)

$\underbrace{\tfrac{1}{2}mv^2}_{\text{kinetic}} = \underbrace{mgy}_{\text{potential}}$ (for some pt between A & B) $\Rightarrow v = \sqrt{2gy}$

$$I[y] \triangleq \int_A^B dt = \int_A^B \frac{ds}{v} = \int_0^l \frac{\sqrt{1 + (y')^2}}{\sqrt{2gy}}\, dx = \frac{1}{\sqrt{2g}} \int_0^l \sqrt{\frac{1 + (y')^2}{y}}\, dx$$

$\therefore$ Our problem is:

$$\min_{\substack{y(x): \\ y(0) = 0,\, y(l) = h}} I[y] = \frac{1}{\sqrt{2g}} \int_0^l \sqrt{\frac{1 + (y')^2}{y}}\, dx \; .$$

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- Fundamental Problem of Calculus of Variations:

Given a Lagrangian: $L: [a,b] \times \mathbb{R} \times \mathbb{R}$, $L(x,y,z)$

admissible functions: $C = \{y: [a,b] \to \mathbb{R} \mid \underbrace{y(a) = y_1,\ y(b) = y_2}_{\text{boundary conditions}},\ y \text{ is } \underbrace{\text{twice differentiable}}_{\text{regularity conditions: Hölder derivatives}}\}$

cost function: $I[y] = \int_a^b L(x, y(x), y'(x))\, dx$ ← called action of physical system

↑ find extremal values

the problem is:

$$\underline{\underline{\min_{y \in C} I[y]}}.$$

- Euler-Lagrange Equations: (Systematic / indirect method of solution)

- Thm: If $y_0 \in C$ minimizes $I[y]$ over C, then:

$$\boxed{\frac{\partial L}{\partial y}(x, y_0, y_0') - \frac{d}{dx}\left(\frac{\partial L}{\partial z}(x, y_0, y_0')\right) = 0}.$$

• only necessary conditions
• solution may be local optimum, or saddle pt, etc.

Fundamental Lemma of Calculus of Variations: (FLCV) $\int_a^b u(x) w(x)\, dx = 0$ for all $w \in C$ if and only if $u(x) = 0$. ← compare with finite case (vectors)

Pf: ($\Rightarrow$) Let $w = u$. $\int_a^b u(x)^2\, dx = 0 \Rightarrow u(x) = 0$. ▨

($\Leftarrow$) Obvious.

} can make this measure theoretic

Pf: Suppose y_0 minimizes $I[y]$ over C. Let $w: [a,b] \to \mathbb{R}$ be any function with $w(a) = w(b) = 0$ and appropriate regularity conditions.

$I[y_0] \leq I[y_0 + tw]$, $\forall t, \forall w: [a,b] \to \mathbb{R}$ ← Perturbation idea

$\left.\frac{d}{dt} I[y_0 + tw]\right|_{t=0} = 0$ ← as y_0 is minimizer

DCT/Leibniz rule to swap diffn & $\int$ (continuity of L & its partial deriv. wrt t)

$$0 = \left.\frac{d}{dt}\int_a^b L(x, y_0 + tw, y_0' + tw')\, dx\right|_{t=0} = \int_a^b \underbrace{\frac{\partial L}{\partial y}(x, y_0 + tw, y_0' + tw') w(x) + \frac{\partial L}{\partial z}(x, y_0 + tw, y_0' + tw') w'(x)}_{= \frac{d}{dt} L(x, y_0 + tw, y_0' + tw')}\, dx \quad \text{[Chain rule]}$$

$t = 0$ plugged in

$$\Rightarrow 0 = \int_a^b \frac{\partial L}{\partial y} w(x) + \underbrace{\left[\frac{\partial L}{\partial z} w(x)\right]_a^b}_{=0, \text{ as } w(a) = w(b) = 0} - \int_a^b \frac{d}{dx}\left[\frac{\partial L}{\partial z}\right] w(x)\, dx \quad \text{[integration by parts]}$$

$$\Rightarrow 0 = \int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial z}\right)\right] w(x)\, dx, \ \forall w$$

▨

$$\overset{\text{FLCV}}{\Rightarrow} \underline{\underline{\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial z}\right) = 0}}$$

- Fermat's Principle of Optics: Light follows the trajectory that takes the shortest time to travel.

$\Rightarrow$ Can use variational calculus & E-L eqns to derive laws of reflection & $\underbrace{\text{refraction}}_{\text{Snell's Law}}$.

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- **Example 1 Solution:** (Shortest Path)

$$L(x,y,z) = \sqrt{1+z^2}\,, \quad \min_{\substack{y(x):\\ y(a)=y_1,\, y(b)=y_2}} I[y] = \int_a^b L(x,y,y')\,dx$$

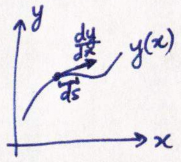
$$\frac{\partial L}{\partial y} = 0\,, \quad \frac{\partial L}{\partial z} = \frac{z}{\sqrt{1+z^2}}$$

Euler-Lagrange equations: $\dfrac{\partial L}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial L}{\partial z}\right) = 0 \Rightarrow \dfrac{d}{dx}\left(\dfrac{y'}{\sqrt{1+(y')^2}}\right) = 0$

$$\Rightarrow \frac{y''}{(1+(y')^2)^{3/2}} = 0 \quad \Rightarrow \underline{\underline{y'' = 0\,,\ y(a) = y_1,\ y(b) = y_2}}$$

So, $y(x)$ is a straight line connecting (a, y_1) and (b, y_2).

- **Remark:** $\dfrac{y''}{(1+(y')^2)^{3/2}}$ is the signed curvature $\left\{\begin{array}{l} \text{- 0 curvature corresponds to lines.} \\ \text{- constant curvature is a circle.}\end{array}\right.$



Clairaut's Thm [cont. second deriv]

$$\underbrace{\frac{d}{ds}\left(\frac{dy}{dx}\right)}_{\text{rate of change of gradient with } ds} \stackrel{\downarrow}{=} \frac{d}{dx}\left(\frac{dy}{ds}\right) = \frac{d}{dx}\left(\frac{dy}{\sqrt{dy^2+dx^2}}\right) = \frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = \underbrace{\frac{y''}{(1+(y')^2)^{3/2}}}_{\text{curvature}}$$

(informal: L not dep on one of its variables $\Rightarrow$ something is constant/some deriv. is 0)

- **Noether's Thm:** Any differentiable symmetry of the action integral (or Lagrangian) has a corresponding conservation law.

① $L = L(x,z)$ independent of y.

$$\underbrace{\frac{\partial L}{\partial y} = 0}_{\text{differential symmetry}} \Rightarrow \text{By Euler-Lagrange eqns, } \frac{d}{dx}\left(\frac{\partial L}{\partial z}\right) = 0 \Rightarrow \underbrace{\boxed{\frac{\partial L}{\partial z} = \text{constant}}}_{\text{conservation law}}$$

eg: (Shortest Path)

$$L(x,y,z) = \sqrt{1+z^2} \Rightarrow \frac{\partial L}{\partial z} = \frac{y'}{\sqrt{1+(y')^2}} = \text{constant} \Rightarrow \underline{y' \text{ constant} \Rightarrow y \text{ linear.}}$$

eg: (Conservation of Linear Momentum)

$$L(t, x, x') = \underbrace{\tfrac{1}{2}m(x')^2}_{\text{kinetic energy}}$$

time, position, velocity

free particle (no potential field)
- minimize total kinetic energy $\int_{t_1}^{t_2} \frac{1}{2}m(x')^2\,dt$ given boundary conds $x(t_1) = x_1$ & $x(t_2) = x_2$

$$\frac{\partial L}{\partial z} = \boxed{m x'(t) = \text{constant}}$$

linear momentum conservation due to symmetry of physical laws in position.

② **Thm:** $L = L(y,z)$ independent of x $\Rightarrow$ BELTRAMI IDENTITY: $\boxed{y'\dfrac{\partial L}{\partial z}(y,y') - L(y,y') = \text{constant}}$

Pf: $\dfrac{d}{dx}\left(y'\dfrac{\partial L}{\partial z} - L\right) = \underbrace{\dfrac{\partial L}{\partial y}y' + \dfrac{\partial L}{\partial z}y''}_{\text{chain rule}} - y''\dfrac{\partial L}{\partial z} - y'\dfrac{d}{dx}\left(\dfrac{\partial L}{\partial z}\right) = y'\underbrace{\left(\dfrac{\partial L}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial L}{\partial z}\right)\right)}_{=0 \text{ by Euler-Lagrange equation}} = 0$ ▨

[continued.]

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② cont. eg: $L(y,z) = \frac{1}{2}mz^2 - V(y)$, $\frac{\partial L}{\partial z} = mz$

$$\Rightarrow \underset{\substack{\uparrow \\ \text{time}}}{L(t}, \underset{\substack{\uparrow \\ \text{pos.}}}{x}, \underset{\substack{\uparrow \\ \text{velocity}}}{x'}) = \underbrace{\tfrac{1}{2}m(x')^2}_{\text{kinetic energy}} - \underbrace{V(x)}_{\text{potential energy}}$$

← Symmetry of physical laws with time

Beltrami: $x' \frac{\partial L}{\partial z}(x,x') - L(x,x') = m(x')^2 - \left(\frac{1}{2}m(x')^2 - V(x)\right) = \text{constant}$

$$\Rightarrow \boxed{\underbrace{\tfrac{1}{2}m(x')^2 + V(x)}_{\text{Hamiltonian = total energy}} = \text{constant}}$$

← energy conservation

- Hamilton's Principle: (The last example motivates Lagrangian mechanics.)

* A system in motion follows a trajectory that minimizes: $\underbrace{\int_{t_1}^{t_2} L(t,x,x')\,dt}_{\text{action integral}}$,

where the Lagrangian $L = \underset{\substack{\uparrow \\ \text{kinetic energy}}}{T} - \underset{\substack{\uparrow \\ \text{potential energy}}}{V}$. (Hamiltonian is $T+V$)

- also called principle of least action (as we minimize action integral)
- solve E-L eqns: $\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial x'}\right) = 0$.

- Example 2 Solution: (Brachistochrone Problem)

$L(x,y,z) = \sqrt{\frac{1+z^2}{y}}$, $\min_{\substack{y(x): \\ y(0)=0,\, y(l)=h}} I[y] = \int_0^l \sqrt{\frac{1+(y')^2}{y}}\,dx$ ← can be derived from Hamilton's Principle after some massaging

↑ remove constant (>0)

L indep. of x ⇒ Beltrami: $y' \frac{\partial L}{\partial z}(y,y') - L(y,y') = \underset{\substack{\uparrow \\ \text{constant}}}{C}$

$$\Rightarrow \frac{(y')^2}{\sqrt{1+(y')^2}\sqrt{y}} - \frac{\sqrt{1+(y')^2}}{\sqrt{y}} = C$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{\frac{k-y}{y}}, \quad k = \frac{1}{C^2} \text{ (constant)}$$

← hard to solve this ODE

Let $\tan(\phi) = \sqrt{\frac{y}{k-y}} \Rightarrow \frac{d\phi}{dx} = \frac{1}{2k\sin^2(\phi)}$ (ϕ func. of x) as $y = k\sin^2(\phi)$ & $\frac{d\phi}{dx} = \frac{d\phi}{dy}\frac{dy}{dx}$.

$\Rightarrow dx = d\phi\, 2k\sin^2(\phi)$ & $dy = d\phi\, 2k\sin(\phi)\cos(\phi)$

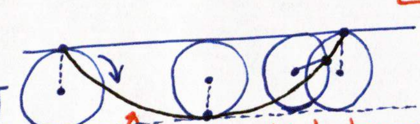
constant of integration

$\Rightarrow x = 2k\int \sin^2(\phi)\,d\phi = 2k\left(\frac{\phi}{2} - \frac{\sin(2\phi)}{4}\right) + c_1$ & $y = \int k\sin(2\phi)\,d\phi = \frac{-k\cos(2\phi)}{2} + c_2$

Boundary condition → $y(0) = 0 \Rightarrow \phi = 0$ & $x = 0 \Rightarrow c_1 = 0$ & $c_2 = \frac{k}{2}$

Let $k = 2a$ & $2\phi = \theta$. Then, $\boxed{\begin{aligned} x &= a(\theta - \sin(\theta)) \\ y &= a(1-\cos(\theta)) \end{aligned}}$. ← Parametric equations of cycloid

- Cycloid:



radius a

path traced by point on rolling circle of radius a

best brachistochrone shape

- solution to brachistochrone problem
- solution to tautochrone problem (same period of oscillation of ball regardless of starting amplitude)
  ↳ Christiaan Huygens

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- **Constrained Optimization:**

- What if we have constraints?

$$\max_{y\in C} \int_a^b L(x,y,y')\,dx \quad \text{subj. to: } \underbrace{\int_a^b K(x,y,y')\,dx = K_0}_{\text{constraint}}$$

$L, K: [a,b]\times\mathbb{R}\times\mathbb{R}$

L ← Lagrangian

$C = \{y:[a,b]\to\mathbb{R} \mid y(a)=y_1,\ y(b)=y_2,\ y \text{ twice differentiable}\}$

- Augmented Lagrangian: $L(x,y,y') + \lambda K(x,y,y')$ ← λ: Lagrange multiplier

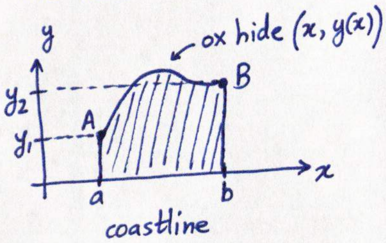
$$I[y] \triangleq \int_a^b L(x,y,y') + \lambda K(x,y,y')\,dx$$

$\max_{y\in C} I[y] \Rightarrow$ Euler-Lagrange Equations: $\left(\frac{\partial L}{\partial y} + \lambda\frac{\partial K}{\partial y}\right) - \frac{d}{dx}\left(\frac{\partial L}{\partial z} + \lambda\frac{\partial K}{\partial z}\right) = 0$,

& explicitly impose $\int_a^b K(x,y,y')\,dx = K_0$

- Example 3: (Dido's Isoperimetric Problem)

Legend is that around 850 B.C., Dido (Queen of Carthage) purchased land from a local king in the North African coastline that could be enclosed by the hide of an ox. The area she enclosed became the city of Carthage.



Want: Max area given arc length of A to B fixed.

$$L(x,y,z) = y, \quad K(x,y,z) = \sqrt{1+z^2}$$

$$\max_{y\in C} \int_a^b L(x,y,y')\,dx = \int_a^b y(x)\,dx \quad \text{area}$$

$$\text{s.t.} \int_a^b K(x,y,y')\,dx = \int_a^b \sqrt{1+(y')^2}\,dx = K_0 \quad \text{fixed length}$$

(K_0 ← length of hide)

equivalent to minimizing arc length for fixed area { Augmented Lagrangian $= L(x,y,y') + \lambda K(x,y,y')$

Euler-Lagrange eqns: $\left(\frac{\partial L}{\partial y} + \lambda\frac{\partial K}{\partial y}\right) - \frac{d}{dx}\left(\frac{\partial L}{\partial z} + \lambda\frac{\partial K}{\partial z}\right) = 0$

$$\Rightarrow 1 - \frac{d}{dx}\left(\lambda\frac{y'}{\sqrt{1+(y')^2}}\right) = 0$$

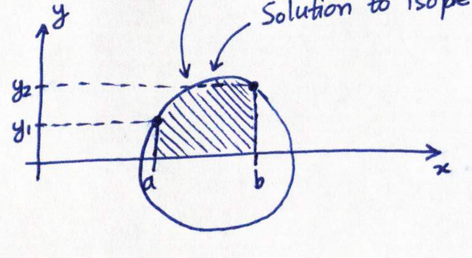
$$\Rightarrow \frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = \frac{1}{\lambda}$$

$$\Rightarrow \frac{y''}{(1+(y')^2)^{3/2}} = \text{constant}$$

Constant curvature ⇒ Solution is circle:

Choose circle so that this length is K_0 & area is maximized

Solution to isoperimetric problem



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